A dynamic Cournot oligopoly model with two time delays

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Table of contents

- Background and Motivation;
- Model construction;
- Results;
- Numerical simulations;
- Conclusions;
- Future research plans.

Background and Motivation

Dynamic Cournot oligopoly games have been widely studied in the literature. In [1] a mixed triopoly game is analyzed, illustrating the competition between a **public firm** and **two private firms**.
Motivated by the mixed competition setting, as well as the delay in the information collection and decision making processes we focus on the analysis of the influence of **the number of private firms** and **the time delays** on the dynamics of a Cournot oligopoly model.
With this aims, we consider **one public firm** and n **private firms** producing differentiated products.

- J. Wang, Z. Bao, J. Huang, Y. Song, Complex Dynamics of Mixed Triopoly Game with Quantity and Price Competition, Discrete Dynamics in Nature and Society, **2021** (2021), ID 9532340, 1–15.

Mathematical Model

Let q_0 be the quantity produced by the public firm with the retail price p_0 and q_i, $i = \overline{1, n}$ the quantity corresponding to the private firm i with the retail price p_i.

The representative consumer maximizes the function [2]:

$$U(q_0, q_1, ..., q_n) - \sum_{i=0}^{n} p_i q_i \ ,$$

where the utility function is supposed to be quadratic and strictly concave, of the form:

$$U(q_0, q_1, ..., q_n) = a \sum_{i=0}^{n} q_i - \frac{b}{2} \left(\sum_{i=0}^{n} q_i^2 + \delta \sum_{i=0}^{n} \sum_{i \neq j} q_i q_j \right) ,$$

with a, b positive parameters and $\delta \in (0, 1)$ the degree of product differentiation.

- Singh, N.; Vives, X. Price and quantity competition in a differentiated duopoly. *RAND J. Econ.* **15** (1984), no. 4, 546–554.

Mathematical Model

The public firm focuses on the maximization of its payoff, that is, the social surplus:

$$SW(q_0, q_1, .., q_n) = a\sum_{i=0}^{n} q_i - \frac{b}{2}\left(\sum_{i=0}^{n} q_i^2 + \delta \sum_{i=0}^{n}\sum_{i\neq j} q_i q_j\right) - \sum_{i=0}^{n} p_i q_i + \sum_{i=0}^{n} P_i, \tag{1}$$

where P_i, $i = \overline{0, n}$ is the profit function of firm i, given by:

$$P_i = (p_i - c_i)q_i, \quad i = \overline{0, n} \tag{2}$$

with c_i the marginal cost of firm i.

The private firm i, $i = \overline{1, n}$ also wants to maximize its payoff, that is, its profit P_i.
We assume that all the private firms have the same marginal costs $c_1 = c_2 = ... = c_n = c$ with $a > c_0 \geq c$ and we obtain:

$$p_i - c - bq_i = 0, \quad i = \overline{1, n}. \tag{3}$$

Mathematical Model

The public firm has bounded rationality and it makes the output's decision based on the expected marginal payoff $\dfrac{\partial SW}{\partial q_0}$. Thus, the dynamical equation of the quantity q_0 is:

$$q_0(t+1) = q_0(t) + \alpha q_0(t)\left[a_0 - bq_0(t) - b\delta\sum_{i=1}^{n} q_i(t)\right],$$

where α is the positive adjustment parameter.
The private firm i, $i = \overline{1,n}$ is naive and its output is established using the reaction function. The dynamical equation of the quantity q_i is:

$$q_j(t+1) = \frac{a_1}{2b} - \frac{\delta}{2}\sum_{i=0, i\neq j}^{n} q_i(t), \, j = \overline{1,n}.$$

Mathematical Model

We consider competitors' time delays in the terms of the bounded rationality, reflected on the ideas presented in [3].

- ⋆ The production of the public firm is adjusted based on the past production levels of the private firms (at time $t - \tau_1$, $\tau_1 > 0$).
- ⋆ The productions of each private firms are updated with respect to the past production (at time $t - \tau_0$, $\tau_0 > 0$) of the public firm.

- A. Elsadany, Dynamics of a delayed duopoly game with bounded rationality, Journal of computational mathematics, **52** (2010), no. 9-10, 1479–1489.

Therefore, we investigate the following nonlinear discrete-time mathematical model with time delays:

$$\begin{cases} q_0(t+1) = q_0(t) + \alpha q_0(t)\left[a_0 - bq_0(t) - b\delta \sum_{i=1}^{n} q_i(t-\tau_1)\right] \\ q_j(t+1) = \dfrac{a_1}{2b} - \dfrac{\delta}{2} q_0(t-\tau_0) - \dfrac{\delta}{2} \sum_{i=1, i\neq j}^{n} q_i(t) \quad , \quad j = \overline{1, n}. \end{cases} \tag{4}$$

Equilibrium points

The discrete dynamical system (4) has two equilibrium points (nonnegative fixed points):

$$E_0 = (0, q^\star, q^\star, ..., q^\star) \qquad \text{and} \qquad E_+ = (q_0^\star, q_1^\star, q_1^\star ..., q_1^\star),$$

where

$$q^\star = \frac{a_1}{b[2 + (n-1)\delta]},$$
$$q_0^\star = \frac{[2 + (n-1)\delta]a_0 - n\delta a_1}{b[2 + (n-1)\delta - n\delta^2]},$$
$$q_1^\star = \frac{a_1 - \delta a_0}{b[2 + (n-1)\delta - n\delta^2]}.$$

Local stability analysis

E_+ is a positive equilibrium point if and only if the following assumptions are satisfied:

$$\begin{aligned} &(\text{A.1}) \qquad [2+(n-1)\delta]a_0 > n\delta a_1 \ , \\ &(\text{A.2}) \qquad a_1 > \delta a_0 \ . \end{aligned}$$

The characteristic equation associated to the liniarized system at one of the equilibrium points $E = (q_0^e, q_1^e, q_1^e, ..., q_1^e) \in \{E_0, E_+\}$ is of the form:

$$\left(\lambda - \frac{\delta}{2}\right)^{n-1} \left[nb\alpha q_0^e \frac{\delta^2}{2} \lambda^{-\tau_0-\tau_1} - \left(\lambda - 1 - \alpha(a_0 - 2bq_0^e - nb\delta q_1^e\right) \left(\lambda + (n-1)\frac{\delta}{2}\right)\right] = 0. \tag{5}$$

The boundary equilibrium E_0

Theorem

If assumption (A.1) holds, the boundary equilibrium point E_0 is a saddle point.

If $q_0^e = 0$ and $q_1^e = q^\star$, the characteristic equation (5) reduces to the equation:

$$\left(\lambda - \frac{\delta}{2}\right)^{n-1}\left(\lambda - 1 - \alpha\frac{(2+(n-1)\delta)a_0 - n\delta a_1}{2+(n-1)\delta}\right)\left(\lambda + (n-1)\frac{\delta}{2}\right) = 0 \quad (6)$$

with $\lambda_1 = 1 + \dfrac{(2+(n-1)\delta)a_0 - n\delta a_1}{2+(n-1)\delta}$ which is positive using assumption (A.1) and $\lambda_0 = \dfrac{\delta}{2} \in (0, 1)$.

The positive equilibrium E_+

For the positive equilibrium, as $q_0^e = q_0^\star$ and $q_1^e = q_1^\star$, the characteristic equation (5) becomes

$$n\beta\frac{\delta^2}{2}\lambda^{-\tau_0-\tau_1} - (\lambda - 1 + \beta)\left(\lambda + (n-1)\frac{\delta}{2}\right) = 0, \tag{7}$$

where

$$\beta = \alpha\frac{[2+(n-1)\delta]a_0 - n\delta a_1}{2+(n-1)\delta - n\delta^2} > 0.$$

The positive equilibrium E_+

Theorem

If assumptions $(A.1)$ and $(A.2)$ hold, when $\tau_0 = \tau_1 = 0$, the equilibrium point E_+ is **asymptotically stable** *if and only if the following inequalities are satisfied:*

$$\delta < \frac{2}{n-1} \qquad \text{and} \qquad \beta < \frac{4 - 2(n-1)\delta}{2 - (n-1)\delta + n\delta^2}. \tag{8}$$

The positive equilibrium E_+

Theorem

For the case of two private firms ($n = 2$), the inequalities (8) reduce to:

$$\beta < \frac{4 - 2\delta}{2 - \delta + 2\delta^2} \qquad \textit{where} \qquad \beta = \alpha \frac{(2+\delta)a_0 - 2\delta a_1}{2 + \delta - 2\delta^2}. \tag{9}$$

Theorem

If assumptions (A.1) and (A.2) hold and the inequalities (8) are satisfied, the equilibrium point E_+ is asymptotically stable for any time delays τ_0 and τ_1.

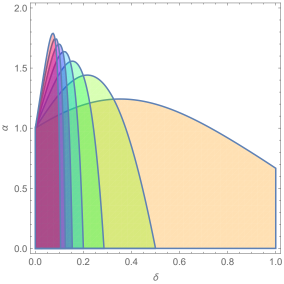

Figure: Stability regions (independent of time delays (τ_0, τ_1)) of the positive equilibrium point E_+ of system (4) in the (δ, α) parameter plane, for different value of $n \in \{2, 5, 8, 11, 14, 17, 20\}$ with $a_0 = 2$ and $a_1 = 2.5$.

Based on previous Theorem, we deduce that the time delays may have a stabilizing effect on the positive equilibrium point E_+, which have been exemplified in Figure. Inequalities (8) provide the delay-independent stability regions of E_+.

Numerical simulations

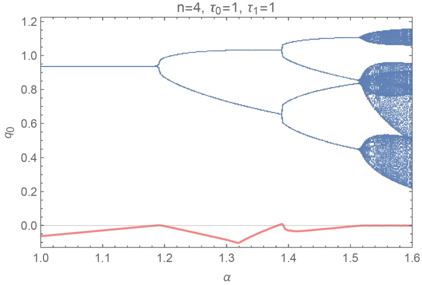

Figure: Bifurcation diagram and largest Lyapunov exponent (shown in red) for system (4) with $n = 4$ private firms and one public firm, with respect to α. Fixed parameter values: $a_0 = 2$, $a_1 = 2.5$, $b = 1$ and $\delta = 0.4$. Time delays: $\tau_0 = \tau_1 = 1$.

A flip bifurcation takes place in a neighborhood of the equilibrium E_+ at the critical value $\alpha^\star$ of the parameter α, which is followed by a period doubling bifurcation at $\alpha \simeq 1.39$ and a Neimark-Sacker bifurcation of the period-4 points at $\alpha \simeq 1.52$.

Numerical simulations

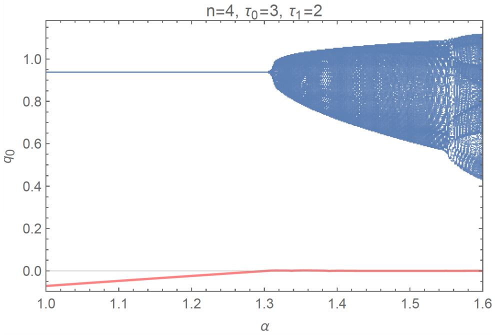

Figure: Bifurcation diagram and largest Lyapunov exponent (shown in red) for system (4) with $n = 4$ private firms and one public firm, with respect to α. Fixed parameter values: $a_0 = 2$, $a_1 = 2.5$, $b = 1$ and $\delta = 0.4$. Time delays: $\tau_0 = 3, \tau_1 = 2$.

A Neimark-Sacker bifurcation takes place at $\alpha \simeq 1.31$, and a stable limit cycle is formed.

Numerical simulations

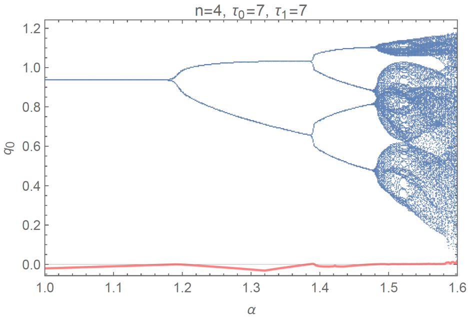

Figure: Bifurcation diagram and largest Lyapunov exponent (shown in red) for system (4) with $n = 4$ private firms and one public firm, with respect to α. Fixed parameter values: $a_0 = 2$, $a_1 = 2.5$, $b = 1$ and $\delta = 0.4$. Time delays: $\tau_0 = \tau_1 = 7$.

A flip bifurcation takes place in a neighborhood of the positive equilibrium which is followed by a period doubling bifurcation at $\alpha \simeq 1.4$.

Phase Portraits

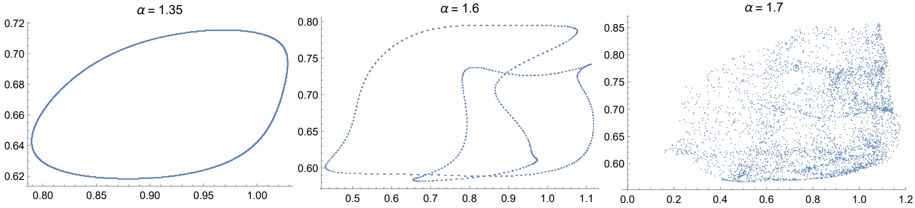

Figure: Phase portraits for system (4) with $n = 4$ private firms and one public firm, for various values of α. Fixed parameter values: $a_0 = 2$, $a_1 = 2.5$, $b = 1$ and $\delta = 0.4$. Time delays: $\tau_0 = 3, \tau_1 = 2$.

Phase Portraits

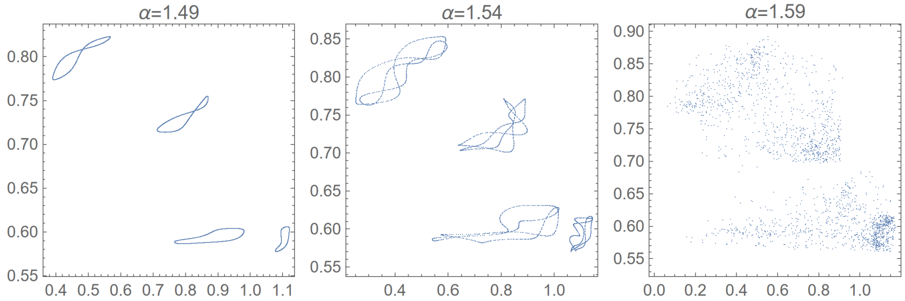

Figure: Phase portraits for system (4) with $n = 4$ private firms and one public firm, for various values of α. Fixed parameter values: $a_0 = 2$, $a_1 = 2.5$, $b = 1$ and $\delta = 0.4$. Time delays: $\tau_0 = \tau_1 = 7$.

Conclusions

- This work generalizes existing findings recently reported in the literature, concerning the stability and bifurcation analysis of a discrete-time Cournot oligopoly game involving one public firm and several private firms;
- Two equilibrium points have been determined and the local stability has been analysed;
- The boundary equilibrium point E_0 is a saddle point;
- The positive equilibrium E_+ is asymptotically stable for certain conditions;
- Numerical simulations reveal complex dynamic behavior and the presence of chaotic regimes for sufficiently large values of the time delays.

Future research plans

Based on theoretical considerations we will extend to other types of Cournot models applied in economics:

- a duopoly model described using discrete equations;
- an extended oligopoly model which the n private firms do not have all-to-all connection.

References I

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