Dynamics of Kerr optical frequency combs

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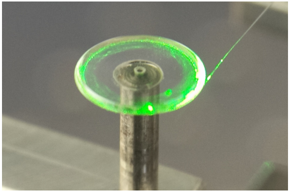
Joint works with: **Mat Johnson, Wesley Perkins, Björn de Rijk**

A problem from optics

A Problem from Optics

- **Home-made whispering gallery modes resonators**



[Yanne Chembo, Rémi Henriet, Aurelien Coillet]

A Problem from Optics

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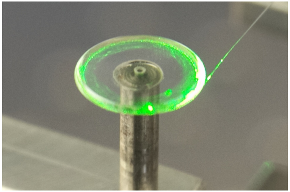

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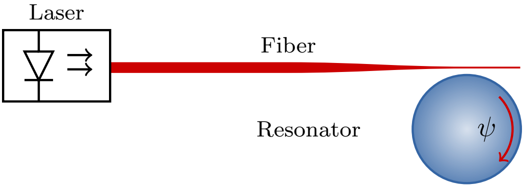
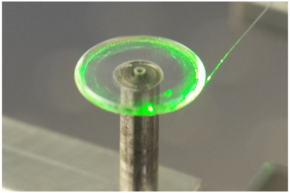


A Problem from Optics

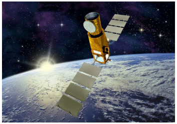
- **Home-made whispering gallery modes resonators**



femto-st SCIENCES & TECHNOLOGIES

[Yanne Chembo, Rémi Henriet, Aurelien Coillet]

- **Applications:** *aerospace engineering*



Clocks

Radars

The Lugiato-Lefever equation (LLE)

[Lugiato & Lefever, 1987]

$$\frac{\partial \psi}{\partial t} = -i\beta \frac{\partial^2 \psi}{\partial x^2} - (1 + i\alpha)\psi + i\psi\,|\psi|^2 + F$$

- $\psi(x,t) \in \mathbb{C}$, $\beta, \alpha \in \mathbb{R}$, $F \in \mathbb{R}$ *(but not only)*
- *NLS-type equation with damping, detuning, and driving*

The Lugiato-Lefever equation (LLE)

[Lugiato & Lefever, 1987]

$$\frac{\partial \psi}{\partial t} = -i\beta \frac{\partial^2 \psi}{\partial x^2} - (1 + i\alpha)\psi + i\psi \left|\psi\right|^2 + F$$

- $\psi(x, t) \in \mathbb{C}$, $\beta, \alpha \in \mathbb{R}$, $F \in \mathbb{R}$ *(but not only)*
- *NLS-type equation with damping, detuning, and driving*
- *extensively studied in the physics literature* [...]
- *few mathematical results ...*

Mathematical model

[Chembo & Menyuk, 2013]

Lugiato-Lefever equation (LLE)

$$\frac{\partial \psi}{\partial t} = -i\beta \frac{\partial^2 \psi}{\partial x^2} - (1 + i\alpha)\psi + i\psi\,|\psi|^2 + F$$

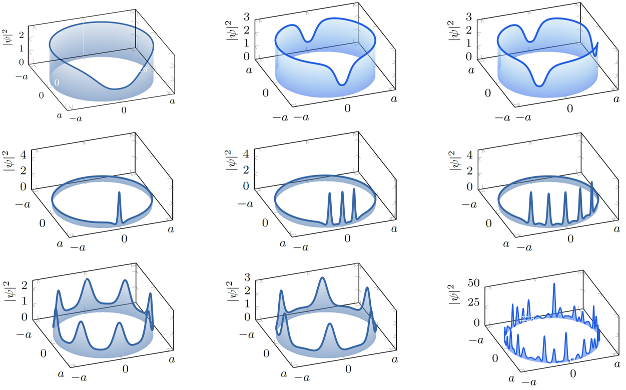
- $\psi(x, t) \in \mathbb{C}$ *intracavity electro-magnetic light field*
- $F > 0$ *external laser pump field intensity*
- $\alpha \in \mathbb{R}$ *frequency detuning between laser and resonator*
- $\beta \in \mathbb{R}$ *resonator dispersion parameter*
  - $\beta > 0$ *normal dispersion*
  - $\beta < 0$ *anomalous dispersion*

Experiments and numerics

- **Frequency combs:** optical signals

  superposition of modes with equally spaced frequencies

  stationary in suitable reference frame.



[Chembo *et al.*, 2014] **[Parra-Rivas, Knobloch, Gomila, . . .]**

Mathematical questions and results

- **existence and stability of nonlinear waves** *(e.g., steady solitons, periodic waves, . . .)*

Mathematical questions and results

- **existence and stability of nonlinear waves** *(e.g., steady solitons, periodic waves, . . .)*

- **not so many results . . .**
  - **existence** *of steady bounded solutions*

    Miyaji, Ohnishi & Tsutsumi (2010)
    Godey, Balakireva, Coillet & Chembo (2014)
    Godey (2016), Delcey & H. (2018)
    Mandel & Reichel (2016), Mandel (2018)

  - **stability** *of steady periodic solutions*

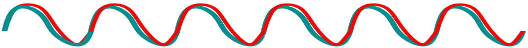
    Miyaji, Ohnishi & Tsutsumi (2011)
    Delcey & H. (2018)
    Hakkaev, Stanislavova, & Stefanov (2018, 2019)
    H., Johnson & Perkins (2021)
    H., Johnson, Perkins & de Rijk (2022)

Stability of periodic waves

Stability of periodic waves

Stability of periodic waves

- **co-periodic perturbations** [period T of the wave]



- **subharmonic perturbations** [period NT, $N \in \mathbb{N}$]

- **localized perturbations**

Stability of periodic waves

Localized perturbations

- *spectral stability*

  [Delcey & H. (2018)]

- *spectral stability implies linear stability*

  [H., Johnson, Perkins (2021)]

- *linear stability implies nonlinear stability*

  [H., Johnson, Perkins, & de Rijk (2022)]

Spectral stability

- **spectrum of the linearized operator** $\mathcal{A}$ [matrix differential operator with periodic coefficients]

$$\boldsymbol{\mathcal{A} = -I + \mathcal{J}\mathcal{L}}$$

$$\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

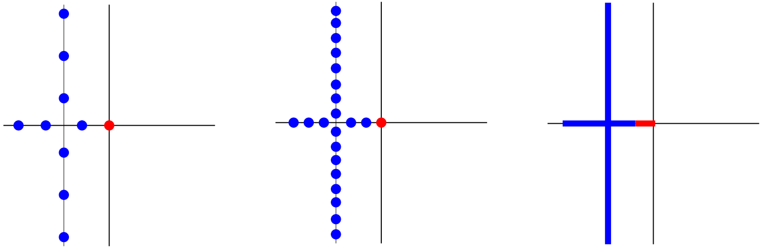
$$\mathcal{L} = \begin{pmatrix} -\beta\partial_x^2 - \alpha + 3\phi_r^2 + \phi_i^2 & 2\phi_r\phi_i \\ 2\phi_r\phi_i & -\beta\partial_x^2 - \alpha + \phi_r^2 + 3\phi_i^2 \end{pmatrix}$$

$\phi = \phi_r + i\phi_i$ denotes the T-periodic wave

Spectral stability

- **spectrum of the linearized operator** $\mathcal{A}$ [matrix differential operator with periodic coefficients]

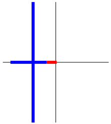
co-periodic | **subharmonic** | **localized**



space: $L^2_{\rm per}(0, NT)$ | space: $L^2_{\rm per}(0, NT)$ | space: $L^2(\mathbb{R})$

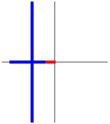
Localized perturbations

- **continuous spectrum**



Localized perturbations

continuous spectrum



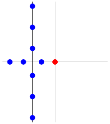
Key tool: **Bloch decomposition**

- **Bloch transform representation** for $g \in L^2(\mathbb{R})$

$$g(x) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} e^{i\xi x}\check{g}(\xi,x)d\xi, \quad \check{g}(\xi,x) := \sum_{\ell\in\mathbb{Z}} e^{2\pi i\ell x/T}\hat{g}(\xi+2\pi\ell/T)$$

- **Bloch operator** $\mathcal{A}_\xi := e^{-i\xi x}\mathcal{A}e^{i\xi x}$ acting in $L^2(0,T)$
- **spectrum**

$$\sigma_{L^2(\mathbb{R})}(\mathcal{A}) = \bigcup_{\xi\in[-\pi/T,\pi/T)} \sigma_{L^2_{\mathrm{per}}(0,T)}(\mathcal{A}_\xi)$$



Diffusive spectral stability

- *the spectrum of the linearized operator $\mathcal{A}$ acting in $L^2(\mathbb{R})$ satisfies*

$$\sigma_{L^2(\mathbb{R})}(\mathcal{A}) \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\} \cup \{0\};$$

- *there exists $\theta > 0$ such that for any $\xi \in [-\pi/T, \pi/T)$ the real part of the spectrum of the Bloch operator $\mathcal{A}_\xi := e^{-i\xi x}\mathcal{A}e^{i\xi x}$ acting in $L^2_{\mathrm{per}}(0, T)$ satisfies*

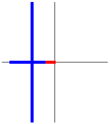
$$\mathrm{Re}\left(\sigma_{L^2_{\mathrm{per}}(0,T)}(\mathcal{A}_\xi)\right) \leq -\theta\xi^2;$$

- *$\lambda = 0$ is a simple eigenvalue of $\mathcal{A}_0$ with associated eigenvector ψ (the derivative ϕ' of the periodic wave).*

- **decay of the C^0-semigroup $e^{\mathcal{A}t}$**

Linear stability

- **decay of the C^0-semigroup $e^{\mathcal{A}t}$**



  - **difficulty:** no spectral gap
  - **Bloch decomposition of the semigroup**

$$e^{\mathcal{A}t}v(x) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} e^{i\xi x}e^{\mathcal{A}_\xi t}\check{v}(\xi,x)d\xi$$

Bloch operator $\mathcal{A}_\xi := e^{-i\xi x}\mathcal{A}e^{i\xi x}$ acting in $L^2_{\rm per}(0,T)$

[Schneider, . . . , Johnson, Noble, Rodrigues, Zumbrun]

Hypotheses

- **diffusive spectral stability;**
- the operator $\mathcal{A}$ generates a C^0-semigroup on $L^2(\mathbb{R})$ and for each $\xi \in [-\pi/T, \pi/T)$ the Bloch operators $\mathcal{A}_\xi$ generate C^0-semigroups on $L^2_{\rm per}(0,T)$;
- there exist positive constants μ_0 and C_0 such that for each $\xi \in [-\pi/T, \pi/T)$ the Bloch resolvent operators satisfy

$$\|(i\mu - \mathcal{A}_\xi)^{-1}\|_{\mathcal{L}(L^2_{\rm per}(0,T))} \leq C_0, \quad \text{for all } |\mu| > \mu_0.$$

checked for LLE: [Delcey, H., 2018], [Stanislavova, Stefanov, 2018]

Main result

- *There exists a constant $C > 0$ such that for any $v \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and all $t > 0$ we have* [1]

$$\left\| e^{\mathcal{A}t} v \right\|_{L^2(\mathbb{R})} \leq C(1+t)^{-1/4} \|v\|_{L^1(\mathbb{R}) \cap L^2(\mathbb{R})}.$$

- *Furthermore,* $e^{\mathcal{A}t} = s_p(t) + \widetilde{S}(t)$ *with*

$$\|s_p(t) v\|_{L^2(\mathbb{R})} \leq C(1+t)^{-1/4} \|v\|_{L^1(\mathbb{R})},$$

$$\left\| \widetilde{S}(t) v \right\|_{L^2(\mathbb{R})} \leq C(1+t)^{-3/4} \|v\|_{L^1(\mathbb{R}) \cap L^2(\mathbb{R})}.$$

[1] The decay is lost when $v \in L^2(\mathbb{R})$, only.

estimates on Bloch semigroups $e^{\mathcal{A}_\xi t}$, $\xi \in [-\pi/T, \pi/T)$

(use: the diffusive spectral stability hypothesis, resolvent estimate, Gearhart-Prüss theorem)

- For any $\xi_0 \in (0, \pi/T)$, there exist $C_0 > 0$, $\eta_0 > 0$, such that

$$\left\| e^{\mathcal{A}_\xi t} \right\|_{\mathcal{L}(L^2_{\rm per}(0,T))} \leq C_0 e^{-\eta_0 t},$$

  for all $t \geq 0$ and all $\xi \in [-\pi/T, \pi/T)$ with $|\xi| > \xi_0$.

- There exists $\xi_1 \in (0, \pi/T)$ and $C_1 > 0$, $\eta_1 > 0$ such that

$$\left\| e^{\mathcal{A}_\xi t} (I - \Pi(\xi)) \right\|_{\mathcal{L}(L^2_{\rm per}(0,T))} \leq C_1 e^{-\eta_1 t},$$

  for all $t \geq 0$ and all $|\xi| < \xi_1$, where $\Pi(\xi)$ is the spectral projection onto the (one-dimensional) eigenspace associated to the eigenvalue $\lambda_c(\xi)$, the continuation for small ξ of the simple eigenvalue 0 of $\mathcal{A}_0$.

Proof

decompose the semigroup $e^{\mathcal{A}t}$ (use: the representation formula for the semigroup and a smooth cut-off function with $\rho(\xi) = 1$ for $|\xi| < \xi_1/2$ and $\rho(\xi) = 0$ for $|\xi| > \xi_1$)

$$e^{\mathcal{A}t}v(x) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \rho(\xi) e^{i\xi x} e^{\mathcal{A}_\xi t} \check{v}(\xi, x) d\xi$$
$$+ \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} (1 - \rho(\xi)) e^{i\xi x} e^{\mathcal{A}_\xi t} \check{v}(\xi, x) d\xi$$
$$=: S_{lf}(t)v(x) + S_{hf}(t)v(x)$$

and show that

$$\|S_{hf}(t)v\|_{L^2(\mathbb{R})} \lesssim e^{-\eta t}\|v\|_{L^2(\mathbb{R})}$$

Proof

- **decompose $S_{lf}(t)v(x)$** (use the diffusive spectral stability hypothesis)

$$
\begin{aligned}
S_{lf}(t)v(x) &= \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \rho(\xi)e^{i\xi x}e^{\mathcal{A}_\xi t}\Pi(\xi)\check{v}(\xi,x)d\xi \\
&\quad + \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \rho(\xi)e^{i\xi x}e^{\mathcal{A}_\xi t}(1-\Pi(\xi))\check{v}(\xi,x)d\xi \\
&=: S_c(t)v(x) + \widetilde{S}_{lf}(t)v(x)
\end{aligned}
$$

and show that

$$
\left\|\widetilde{S}_{hf}(t)v\right\|_{L^2(\mathbb{R})} \lesssim e^{-\eta t}\|v\|_{L^2(\mathbb{R})}
$$

PROOF

- **decompose** $S_c(t)v(x)$ (use formula for $\Pi(\xi)$)

$$S_c(t)v(x) = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \rho(\xi)e^{i\xi x}e^{\mathcal{A}_\xi t}\Pi(0)\check{v}(\xi,x)d\xi$$
$$+\frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \rho(\xi)e^{i\xi x}e^{\mathcal{A}_\xi t}(\Pi(0)-\Pi(\xi))\check{v}(\xi,x)d\xi$$
$$=: s_p(t)v(x) + \widetilde{S}_c(t)v(x)$$

and show that[2]

$$\left\|\widetilde{S}_c(t)v\right\|_{L^2(\mathbb{R})} \lesssim \|\xi e^{-d\xi^2 t}\|_{L^2_\xi(\mathbb{R})}\|v\|_{L^1(\mathbb{R})} \lesssim (1+t)^{-3/4}\|v\|_{L^1(\mathbb{R})}$$

$$\|s_p(t)v\|_{L^2(\mathbb{R})} \lesssim \|e^{-d\xi^2 t}\|_{L^2_\xi(\mathbb{R})}\|v\|_{L^1(\mathbb{R})} \lesssim (1+t)^{-1/4}\|v\|_{L^1(\mathbb{R})}$$

[2] The decay is lost when $v \in L^2(\mathbb{R})$, only.

Nonlinear stability

- **linear stability implies nonlinear stability**

Nonlinear stability

linear stability implies nonlinear stability

$\rightsquigarrow$ rely on Duhamel's formulation and properties of the semigroup

$\rightsquigarrow$ **two main difficulties:**

- semigroup with slow decay $(1+t)^{-1/4}$
- C^0-semigroup

Nonlinear stability

First difficulty: semigroup with slow decay $(1+t)^{-1/4}$

- no decay for the (unmodulated) perturbation

$$\boxed{\tilde{v}(x,t) = \psi(x,t) - \phi(x)}$$

satisfying (Duhamel formulation)

$$\tilde{v}(t) = e^{\mathcal{A}t}v_0 + \int_0^t e^{\mathcal{A}(t-s)}\widetilde{\mathcal{N}}(\tilde{v}(s))\,ds$$

Nonlinear stability

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$$\tilde{v}(t) = e^{\mathcal{A}t} v_0 + \int_0^t e^{\mathcal{A}(t-s)} \widetilde{\mathcal{N}}(\tilde{v}(s))\, ds$$

- define a **modulated perturbation**

$$\boxed{v(x,t) = \psi(x - \gamma(x,t), t) - \phi(x)}$$

[Schneider, Doelman, Sandstede, Scheel, Uecker, . . . Johnson, Noble, Rodrigues, Zumbrun]

Nonlinear stability

modulated perturbation

$$\boldsymbol{v(x,t) = \psi(x - \gamma(x,t), t) - \phi(x)}$$

$\rightsquigarrow$ satisfies $(\partial_t - \mathcal{A})(v + \gamma\phi') = \mathcal{N}(v, \gamma, \partial_t\gamma) + (\partial_t - \mathcal{A})(\gamma_x v)$

Nonlinear stability

modulated perturbation

$$v(x,t) = \psi(x - \gamma(x,t), t) - \phi(x)$$

⇝ satisfies $(\partial_t - \mathcal{A})(v + \gamma\phi') = \mathcal{N}(v, \gamma, \partial_t\gamma) + (\partial_t - \mathcal{A})(\gamma_x v)$

⇝ use Duhamel formulation and $e^{\mathcal{A}t} = s_p(t) + \widetilde{S}(t)$ to:

- define the **phase modulation $\gamma(x,t)$**

$$\gamma(t) = s_p(t)v_0 + \int_0^t s_p(t-s)\mathcal{N}(v(s), \gamma(s), \partial_t\gamma(s))\, ds$$

(such that it captures the slowest decay rate $(1+t)^{-1/4}$)

- obtain a formula for $v(x,t)$

$$v(t) = \widetilde{S}(t)v_0 + \int_0^t \widetilde{S}(t-s)\mathcal{N}(v(s), \gamma(s), \partial_t\gamma(s))\, ds + \gamma_x(t)v(t)$$

(stronger decay rate $(1+t)^{-3/4}$; enough to conclude ...)

Second difficulty: C^0-semigroup

- no control of derivatives of the modulated perturbation

$$\boxed{v(x,t) = \psi(x-\gamma(x,t),t) - \phi(x)}$$

appearing in the nonlinear terms $\mathcal{N}(v(s),\gamma(s),\partial_t\gamma(s))$

NONLINEAR STABILITY

Second difficulty: C^0-semigroup

- no control of derivatives of the modulated perturbation

$$\boxed{\mathbf{v(x,t) = \psi(x - \gamma(x,t), t) - \phi(x)}}$$

appearing in the nonlinear terms $\mathcal{N}(v(s), \gamma(s), \partial_t \gamma(s))$

- use **integration by parts** to gain derivatives and decay in the formula for the phase modulation $\boldsymbol{\gamma(x,t)}$

$$\boxed{\boldsymbol{\gamma(t)} = s_p(t)v_0 + \int_0^t s_p(t-s)\mathcal{N}(v(s), \gamma(s), \partial_t \gamma(s))\, ds}$$

Nonlinear stability

Second difficulty: C^0-semigroup

- no control of derivatives of the modulated perturbation

$$v(x,t) = \psi(x-\gamma(x,t),t) - \phi(x)$$

appearing in the nonlinear terms $\mathcal{N}(v(s),\gamma(s),\partial_t\gamma(s))$

- use **integration by parts** to gain derivatives and decay in the formula for the phase modulation $\gamma(x,t)$
- also use the **unmodulated perturbation**

$$\tilde{v}(x,t) = \psi(x,t) - \phi(x)$$

(slow decay but no loss of derivatives)

[Sandstede & de Rijk (2021)]

Nonlinear stability

- for the unmodulated perturbation $\tilde{v}(x,t)$ and the modulated perturbation $v(x,t)$
  - obtain the decay rate $(1+t)^{-3/4}$ for the modulated perturbation [3]

$$v(t) = \widetilde{S}(t)v_0 + \int_0^t \widetilde{S}(t-s)\mathcal{N}(v(s),\gamma(s),\partial_t\gamma(s))\,ds + \gamma_x(t)v(t)$$

  - obtain the needed regularity for the unmodulated perturbation

$$\tilde{v}(t) = e^{\mathcal{A}t}v_0 + \int_0^t e^{\mathcal{A}(t-s)}\widetilde{\mathcal{N}}(\tilde{v}(s))\,ds$$

  - use **mean value inequalities** to connect $\tilde{v}(x,t)$ and $v(x,t)$

[3] Recall the decay rates in the decomposition $e^{\mathcal{A}t} = s_p(t) + \widetilde{S}(t)$

Main result

- *There exist constants* $\varepsilon, M > 0$ *such that, whenever the initial perturbation* $v_0 \in L^1(\mathbb{R}) \cap H^4(\mathbb{R})$ *satisfies* $\boldsymbol{E_0} := \|\boldsymbol{v_0}\|_{L^1 \cap H^4} < \varepsilon$, *there exist functions*

$$\tilde{\boldsymbol{v}}, \gamma \in C\big([0,\infty), H^4(\mathbb{R})\big) \cap C^1\big([0,\infty), H^2(\mathbb{R})\big),$$

with $\tilde{v}(0) = v_0$ *and* $\gamma(0) = 0$ *such that* $\psi(t) = \phi + \tilde{v}(t)$ *is the unique global solution of LLE with initial condition* $\psi(0) = \phi + v_0$.

- *The inequalities*

$$\|\psi(t) - \phi\|_{L^2},\ \|\gamma(t)\|_{L^2} \leq ME_0(1+t)^{-\frac{1}{4}},$$

$$\|\psi(\cdot - \gamma(\cdot, t), t) - \phi\|_{L^2}, \leq ME_0(1+t)^{-\frac{3}{4}},$$

hold for all $t \geq 0$.

Further issues

- *stability of solitary and generalized solitary waves*
- *existence and stability of other observed solutions: multi-solitons, breathers, ...*
- *other versions of LLE (non-constant source term F, two spatial dimensions, ...)*
- *connections between mathematical and experimental results*