Shape Spaces of Nonlinear Flags

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Motivation

Shape Analysis

Objective: measuring deformations of shapes independently of the way shapes are parameterized.

Tools needed: a Riemannian metric that is invariant to reparameterizations and quantifies the degrees of bending and stretching of a shape.

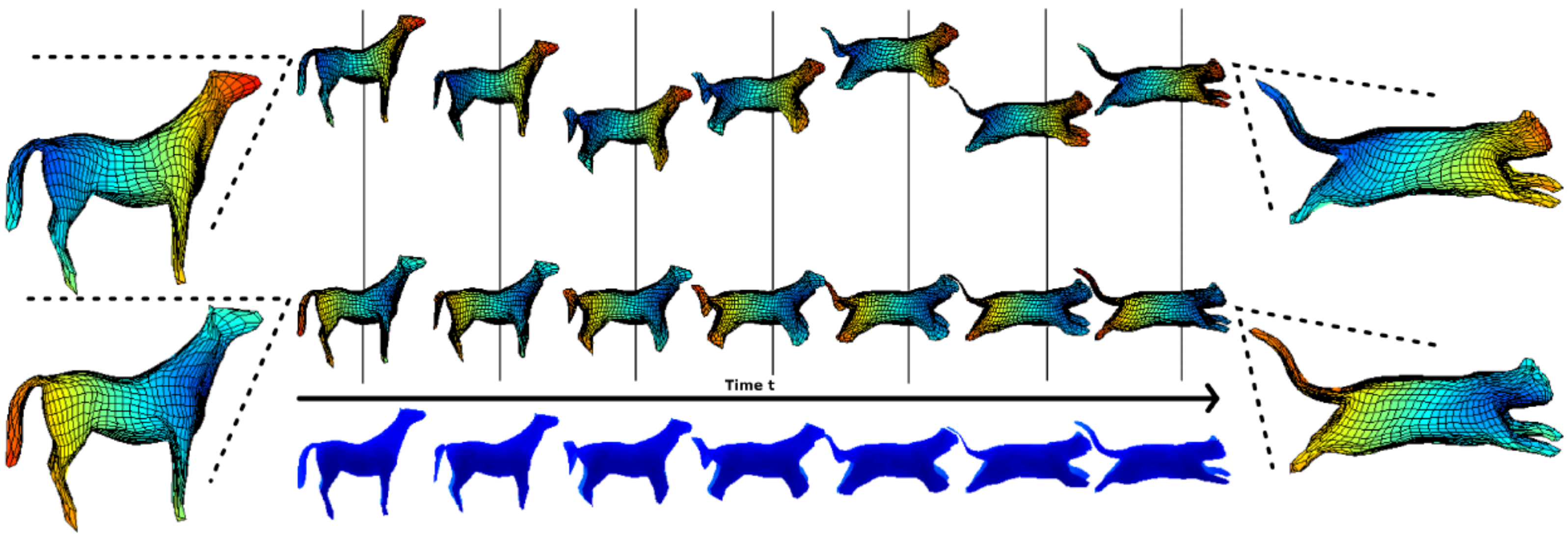

Fig 1. Two paths of transformations consisting of the same shapes with different parameterizations

The Spaces Under Consideration

Nonlinear Flag Manifolds

Let M be a manifold and $\mathcal{S} = (S_1, ..., S_r)$ a collection of closed manifolds.

The nonlinear flag manifold of type $\mathcal{S}$ is the Fréchet manifold of nested submanifolds of M,

$$\mathrm{Flag}_{\mathcal{S}}(M) = \left\{ (N_1, ..., N_r) \in \prod_{i=1}^{r} \mathrm{Gr}_{S_i}(M) : N_i \subset N_{i+1} \right\},$$

where

$$\mathrm{Gr}_{S_i}(M) = \{ N \subseteq M : N \text{ is diffeomorphic to } S_i \}$$

is the nonlinear Grassmanian of type S_i in M.

Principal bundle

Let $(\mathcal{S},\iota) : S_1 \overset{\iota_1}{\hookrightarrow} S_2 \overset{\iota_2}{\hookrightarrow} ... \overset{\iota_{r-1}}{\hookrightarrow} S_r$ be a sequence of embeddings.

The Frechet-Lie subgroup $G \subset \mathrm{Diff}(S_r)$ of diffeomorphisms compatible with the embeddings

$$G = \{\gamma \in \mathrm{Diff}(S_r) : \gamma \circ \iota_{r-1} \circ \ldots \circ \iota_i = \iota_{r-1} \circ \ldots \circ \iota_i \circ \bar{\gamma}_i,\ \ \bar{\gamma}_i \in \mathrm{Diff}(S_i)\}$$

is the structure group of a principal bundle:

$$\pi : \mathrm{Emb}(S_r, M) \to \mathrm{Flag}_{\mathcal{S},\iota}(M), \quad \pi(F) = (f_1(S_1), \ldots, f_r(S_r)),$$

where $f_i = F \circ \iota_{r-1} \circ \ldots \circ \iota_i \in \mathrm{Emb}(S_i, M)$.

Here the base manifold $\mathrm{Flag}_{\mathcal{S},\iota}(M)$, called the nonlinear flag manifold of type $(\mathcal{S},\iota)$, is a union of connected components of $\mathrm{Flag}_{\mathcal{S}}(M)$.

Fig 2. Examples of elements in the space of nonlinear flags.

The (Pre-)Shape Space

Consider $M = \mathbb{R}^3$ and the pair $\mathcal{S} = (\mathbb{S}^1, \mathbb{S}^2)$ consisting of the unit sphere $\mathbb{S}^2$ and the unit circle $\mathbb{S}^1$ embedded at the equator:

$$\iota : \mathbb{S}^1 \hookrightarrow \mathbb{S}^2$$

Let $F : \mathbb{S}^2 \to \mathbb{R}^3$ be an embedding. We will denote its image by $\Sigma := F(\mathbb{S}^2)$ and the image of its restriction to the equator by $C := (F \circ \iota)(\mathbb{S}^1)$. We call the space of all possible parameterizations of the pair (C, Σ) the pre-shape space:

$$\mathcal{P} = \mathsf{Emb}(\mathbb{S}^2, \mathbb{R}^3)$$

We call the space of all unparameterzied pairs (C, Σ) the shape space:

$$\mathcal{F} := \mathsf{Flag}_{\mathcal{S},\iota}(\mathbb{R}^3)$$

We have the principal bundle

$$\pi : \mathcal{P} \to \mathcal{F}, \quad \pi(F) = (f(\mathbb{S}^1), F(\mathbb{S}^2)) \text{ where } f := (F \circ \iota)$$

with structure group

$$G = \left\{ \gamma \in \mathsf{Diff}(\mathbb{S}^2) : \gamma \circ \iota = \iota \circ \bar{\gamma},\ \bar{\gamma} \in \mathsf{Diff}(\mathbb{S}^1) \right\}$$

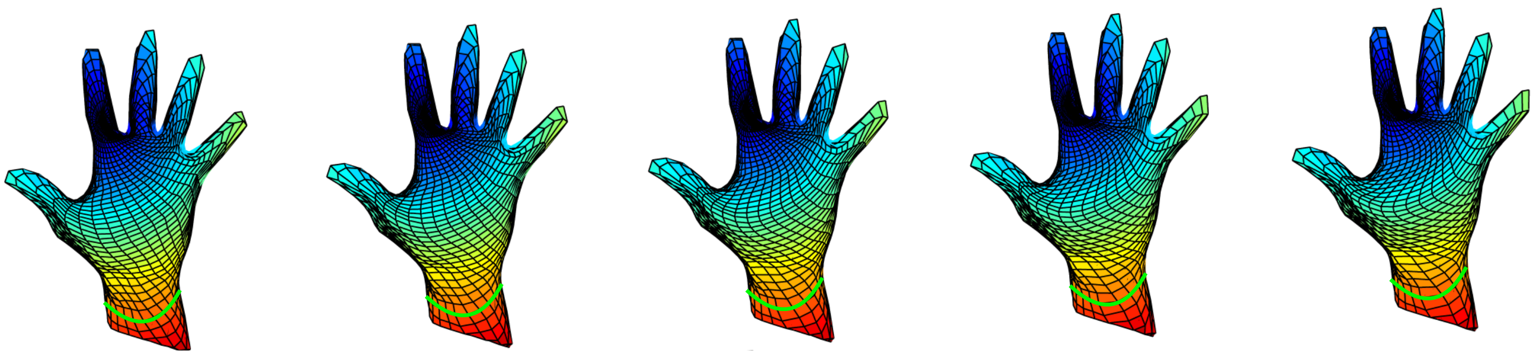

Fig 3: Examples of elements in the same orbit under the group of reparameterizations.

Tangent Spaces

Vertical / Normal Bundle

The kernel of $T\pi$ is called the vertical space and represents the tangent space to the orbit of $F \in \mathcal{P}$ under the action of the group G:

$$\text{Ver}_F = \{X_F \in T_F\mathcal{P} : X_F \circ F^{-1} \text{ is tangent to } \Sigma \\ \text{and its restriction to } C \text{ is tangent to } C\}$$

The normal bundle Nor is the vector bundle over the pre-shape space $\mathcal{P}$, whose fiber over an embedding F is the following quotient vector space:

$$\text{Nor}_F := T_F\mathcal{P}/\,\text{Ver}_F$$

The tangent bundle to the shape space $\mathcal{F}$ can be identified with the following quotient space:

$$T\mathcal{F} = \text{Nor}\,/G$$

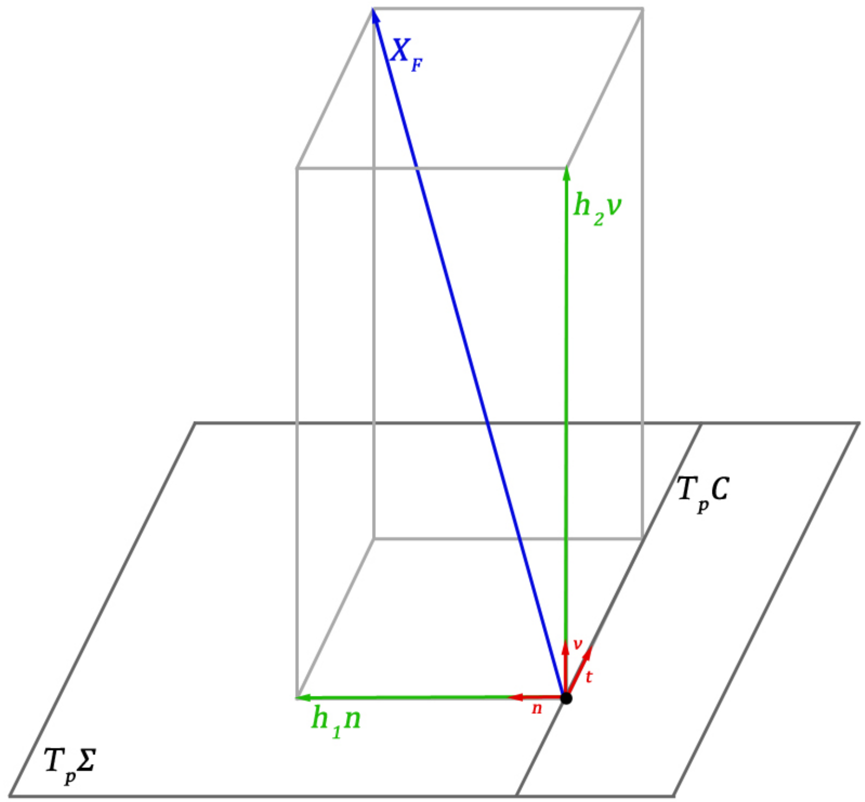

Fig 4. Deformation vector field and Darboux frame (t, n, ν), where ν is the unit normal vector field on the oriented surface Σ, t is the unit vector field tangent to the oriented curve C and $n := \nu \times t$ is the unit normal to the curve C contained in the tangent space to the surface.

Tangent Space to the Shape Space

Let F be a parameterization of (C, Σ). Consider the linear surjective map

$$\Psi_F \ : T_F\mathcal{P} \simeq \mathcal{C}^\infty(\mathbb{S}^2, \mathbb{R}^3) \to \mathcal{C}^\infty(C) \times \mathcal{C}^\infty(\Sigma), \tag{1}$$

which maps $X_F \in T_F\mathcal{P}$ to (h_1, h_2) defined by

$$\begin{aligned} h_1 &:= \langle (X_F \circ \iota) \circ (F \circ \iota)^{-1}, n \rangle \in \mathcal{C}^\infty(C), \\ h_2 &:= \langle X_F \circ F^{-1}, \nu \rangle \in \mathcal{C}^\infty(\Sigma). \end{aligned} \tag{2}$$

Then the kernel of Ψ_F is the vertical subspace Ver_F, hence Ψ_F defines a map from the quotient space $\text{Nor}_F = T_F\mathcal{P}/\text{Ver}_F$ into $\mathcal{C}^\infty(C) \times \mathcal{C}^\infty(\Sigma)$. The resulting bundle map Ψ is G-invariant providing an isomorphism between the tangent space $T_{(C,\Sigma)}\mathcal{F}$ and $\mathcal{C}^\infty(C) \times \mathcal{C}^\infty(\Sigma)$.

Elastic Metrics

Riemannian Metrics on Shape Space I

1. We embed our shape space $\mathcal{F}$ of surfaces decorated with curves in the Cartesian product $\mathrm{Gr}_{\mathbb{S}^1}(\mathbb{R}^3) \times \mathrm{Gr}_{\mathbb{S}^2}(\mathbb{R}^3)$, where $\mathrm{Gr}_{\mathbb{S}^1}(\mathbb{R}^3)$ denotes the shape space of curves and $\mathrm{Gr}_{\mathbb{S}^2}(\mathbb{R}^3)$ the shape space of surfaces.
2. We choose a family $g^{a,b}$ of $\mathrm{Diff}^+(\mathbb{S}^1)$-invariant metrics on the space of parameterized curves which gives us the following squared norm of the variation δf:

$$\mathcal{E}'_f(\delta f) := g_f^{a,b}(\delta f, \delta f) = a \int_{\mathbb{S}^1} \left(\frac{\delta r}{r}\right)^2 d\ell + b \int_{\mathbb{S}^1} |\delta t|^2 d\ell$$

where $r = \|\dot{f}(t)\|$.
3. The family $g^{a,b}$ defines a family of Riemannian metrics on the shape space of curves by restricting to the normal variations of curves $\delta f = h_1 n + (h_2|_C)\nu$

$$\mathcal{E}'_F(h_1 n + (h_2|_C)\nu) = a \int_C (h_1 \kappa_g + h_2|_C \kappa_n)^2 d\ell + b \int_C (D_s h_1 - h_2|_C \tau_g)^2 d\ell$$
$$+ c \int_C (D_s(h_2|_C) + h_1 \tau_g)^2 d\ell$$

where $D_s h(t) = \frac{\dot{h}(t)}{\|\dot{f}(t)\|}$ is the arc-length derivative of the variation h.

Riemannian Metrics on Shape Space II

4. We choose a family $g^{a,b,c}$ of $\mathrm{Diff}^+(\mathbb{S}^2)$-invariant metrics on the space of parameterized surfaces which gives us the following squared norm of the variation δF

$$\mathcal{E}''_F(\delta F) := a\int_{\mathbb{S}^2} \mathrm{Tr}((g^{-1}\delta g)_0)^2 dA + b\int_{\mathbb{S}^2}(\mathrm{Tr}(g^{-1}\delta g))^2 dA + c\int_{\mathbb{S}^2}|\delta\nu|^2 dA$$

where A_0 is the traceless part of the matrix A.

5. The family $g^{a,b,c}$ defines a family of Riemannian metrics on the shape space of sufaces by restricting to the normal variations of surfaces $\delta F = h_2\nu$:

$$\begin{aligned}\mathcal{E}''_F(h_2\nu) = a\int_\Sigma (h_2)^2(\kappa_1-\kappa_2)^2 dA + b\int_\Sigma (h_2)^2(\kappa_1+\kappa_2)^2 dA \\ + c\int_\Sigma |\nabla h_2|^2 dA\end{aligned}$$

6. The product of these metrics is then restricted to $\mathcal{F}$ using the characterization of the tangent space to $\mathcal{F}$ given in Theorem 1.

Riemannian Metrics on Manifolds of Decorated Surfaces

The gauge invariant elastic metrics for parameterized curves respectively surfaces lead to a 6-parameter family of Riemannian metrics on the shape space of embedded surfaces decorated with curves:

$$\begin{aligned}\mathcal{G}_{(C,\Sigma)}(h_1,h_2) = a_1 \int_C (h_1\kappa_g + h_2|_C\kappa_n)^2 d\ell \quad &+ a_2 \int_\Sigma (h_2)^2(\kappa_1-\kappa_2)^2 dA \\ + b_1 \int_C (D_s h_1 - h_2|_C \tau_g)^2 d\ell \quad &+ b_2 \int_\Sigma (h_2)^2(\kappa_1+\kappa_2)^2 dA \\ + c_1 \int_C (D_s(h_2|_C) + h_1\tau_g)^2 d\ell \quad &+ c_2 \int_\Sigma |\nabla h_2|^2 dA\end{aligned}$$

for $h_1 \in \mathcal{C}^\infty(C)$ and $h_2 \in \mathcal{C}^\infty(\Sigma)$.

We call this an elastic metric because it quantifies the deformations of a shape. In our case, the term a_1 measures the stretching of the curve, the terms b_1 and c_1 measure the bending of the curve, the terms a_2 and b_2 measure the stretching of the surface and the term c_2 measures the bending of the surface.

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