The Geometry of p-adic numbers

What is a p-adic number? $p \geq 2$ a prime number

- $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$ on $\mathbb{Q}$.
- $\forall x \in \mathbb{Q}_p$ has a unique p-adic decomposition

$$x = \underbrace{\underset{0 \neq}{a_{-n}}\, p^{-n} + a_{-n+1}\, p^{-n+1} + \ldots +}_{\text{finite sum}} \underbrace{a_0 + a_1 \cdot p + a_2 \cdot p^2 + \ldots}_{\text{can be an infinite sum}} ; \quad a_i \in \{0, \ldots, p-1\} ; \quad |x|_p = p^{n}$$

- $\mathbb{Z}_p := \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$; $x \in \mathbb{Z}_p$ is written: $x = a_0 + a_1 p + a_2 p^2 + \ldots$
- $\mathbb{Z}_p$ is compact and open, with respect to the p-adic norm $|\cdot|_p$ topology.
- $\mathbb{Q}_p$ is a totally disconnected field, $\mathbb{Q} \subseteq \mathbb{Q}_p$

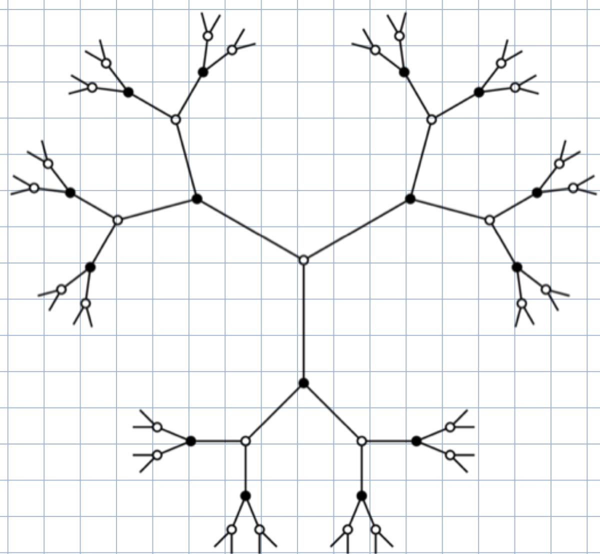
How do we study properties of $SL(n, \mathbb{Q}_p)$?

- $SL(n, \mathbb{Q}_p) =$ $n \times n$ matrices with entries in $\mathbb{Q}_p$ of determinant 1.
- $SL(n, \mathbb{Q}_p)$ is a totally disconnected locally compact group (i.e. the only connected components are the elements $x \in SL(n, \mathbb{Q}_p)$) ($\mathbb{Q}_p$ is totally disconnected).
- Generally, properties of L.C.G are studied via their action on canonically associated metric spaces.

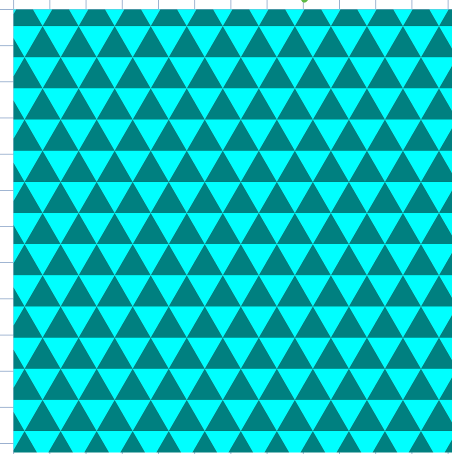
Examples: $SL(n, \mathbb{R})$ acts on the Riemannian manifold $SL(n, \mathbb{R}) / SO(n)$ - called symmetric space of $SL(n, \mathbb{R})$.

- For $SL(n, \mathbb{Q}_p)$ we also have an associated "symmetric space". This is a simplicial complex called the Bruhat-Tits building of $SL(n, \mathbb{Q}_p)$.
- For $SL(2, \mathbb{Q}_p)$, the Bruhat-Tits building is T_{p+1}, i.e. a $(p+1)$-regular tree.
- Symmetric spaces, Bruhat-Tits buildings, regular trees are CAT(0) spaces, non-positively curved.

The Bruhat-Tits building of $SL(2, \mathbb{Q}_2)$



Part of the Bruhat-Tits building of $SL(3, \mathbb{Q}_p)$



The B-T building of $SL(n, \mathbb{Q}_p)$ is a union of tessellated $\mathbb{R}^{n-1}$ spaces (with simplices of some "type", that are glued along faces, ..., vertices of the maximal simplices called chambers and following some axioms.

Up to conjugacy, some examples of important closed subgps of $SL(n, \mathbb{Q}_p)$

Borel $\quad B = \left\{ \begin{pmatrix} * & & * \\ & * & \\ 0 & & \ddots \; * \end{pmatrix} \,\middle|\, * \in \mathbb{Q}_p \right\} =$ upper triangular matrices of $SL(n, \mathbb{Q}_p)$

"general parabolic" $\quad P = \left\{ \begin{pmatrix} \boxed{A_1} & & * \\ & \boxed{A_2} & \\ 0 & & \ddots \; \boxed{A_k} \end{pmatrix} \,\middle|\, * \in \mathbb{Q}_p,\ A_i \text{ block matrices} \right\} =$ block-upper triangular matrices of $SL(n, \mathbb{Q}_p)$.

"good maximal compact" $\quad SL(n, \mathbb{Z}_p)$.

Cartan subgroup $\quad \mathrm{Diag} := \left\{ \begin{pmatrix} a_1 & & 0 \\ & a_2 & \\ 0 & & \ddots \; a_n \end{pmatrix} \,\middle|\, a_i \in \mathbb{Q}_p^*,\ \prod_{i=1}^{n} a_i = 1 \right\}$ - closed subgroup, maximal tori.

The action of $SL(n,\mathbb{Q}_p)$ on its B-T building, topology, subgroups.

- As B-T of $SL(n,\mathbb{Q}_p)$ is a proper CAT(0) space, one can define its "visual boundary".
- The visual boundary of B-T has a structure of a building that is spherical, called the Tits building associated with $SL(n,\mathbb{Q}_p)$.
- For σ a simplex of the sph-T, $G_\sigma := \{g \in SL(n,\mathbb{Q}_p) \mid g(\sigma)=\sigma\}$ - parabolic subgp of $SL(n,\mathbb{Q}_p)$
- The Borel subgp of $SL(n,\mathbb{Q}_p)$, the minimal parabolic.
- The stabilizer in $SL(n,\mathbb{Q}_p)$ of a simplex (e.g vertex) of B-T is compact and open
- For $x \in$ B-T a vertex, $G_x := \{g \in SL(n,\mathbb{Q}_p) \mid g(x)=x\}$ is maximal compact
  $SL(n,\mathbb{Z}_p)$ is an example of such maximal compact subgroup.
- $SL(n,\mathbb{Q}_p)$ is endowed with the compact-open topology inherited from B-T.

What is the "topology" in order to study limits of subgroups?

Chabauty topology (1950): he proved that appropriate sets of lattices of some locally compact groups are relatively compact.

- It is defined for a locally compact topological space X; $\mathcal{F}(X)$ the set of closed subsets of X.

Proposition - Definition (Chabauty topology).

Take X a locally compact metric space. A sequence of closed subsets $\{F_n\}_{n\geq 0} \subset \mathcal{F}(X)$ converges to $F \in \mathcal{F}(X)$ if and only if the following are true:

1) for every $f \in F$ there is a sequence $\{f_n \in F_n\}$ converging to f;
2) for every sequence $\{f_n \in F_n\}_{n\geq 0}$ if there is a strictly increasing subsequence $\{n_k\}_{k\in\mathbb{N}}$ s.th $\{f_{n_k} \in F_{n_k}\}_{k\in\mathbb{N}}$ converges to f, then $f \in F$.

- The space $\mathcal{F}(X)$ is compact with respect to the Chabauty topology.
- For G a locally compact group (LCG), $S(G)$ the set of all closed subgps of G. $S(G)$ compact

Chabauty limits of certain families of closed subgroups of $SL(n,\mathbb{Q}_p)$.

Recall, $S(G)$ set of all closed subgps of a loc. cpt. gp G is compact w.r.t. the Chabauty top.

- Take $G := SL(n,\mathbb{Q}_p)$

$$\{ G_x := \{ g \in SL(n,\mathbb{Q}_p) \mid g(x) = x \} \}_{x \text{ point in } B\text{-}T}$$

$$Cart(G) := \{ g \, Diag \, g^{-1} \mid g \in SL(n,\mathbb{Q}_p) \}$$

Questions: What are $\overline{\{G_x\}_{x\in B\text{-}T}}^{Ch}$ and $\overline{Cart(G)}^{Ch}$?

These are joint works with A. Leitner & A. Valette, published in 2021 / 2022

$$\overline{\{G_x\}_{x \in B\text{-}T}}^{Ch} = ?$$

Brief answer:
- The "Chabauty compactification" of $\{G_x\}_{x\in B\text{-}T}$ corresponds to the compactification of the $B\text{-}T$ building of $SL(n,\mathbb{Q}_p)$ given by the spherical = Tits building on the visual boundary.
- In particular, $H \in \overline{\{G_x\}_x}^{Ch} \setminus \{G\}_x$ is, up to conj, the elliptic part of a parabolic subgroup $\begin{pmatrix} \boxed{A_1} & & * \\ & \boxed{A_2} & \\ & & \ddots \; \boxed{A_k} \end{pmatrix}$

A more general result was proven by Guivarc'h & Rémy in '06 using proba methods.

$\overline{\mathrm{Cart}(G)}^{Ch} = ?$

$\mathrm{Cart}(G) := \{ g \,\mathrm{Diag}\, g^{-1} \mid g \in SL(n, \mathbb{Q}_p) \}$, $\mathrm{Diag} = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \right\}$ in $SL(n, \mathbb{Q}_p)$.

1) The diagonal Cartan $\mathrm{Diag} \leq SL(n, \mathbb{Q}_p)$ corresponds to an apartment of B-T building

2) Diag is obviously abelian $\Rightarrow$ all groups of $\overline{\mathrm{Cart}(G)}^{Ch}$ are abelian.

3) By 1)+2), up to conjugacy in $SL(n, \mathbb{Q}_p)$, $\forall H \in \overline{\mathrm{Cart}(G)}^{Ch} \Rightarrow H \leq B$ - Borel

$\Rightarrow H \in \overline{\mathrm{Cart}(G)}^{Ch}$:
- either contains elliptic elements, H called elliptic limit
- or contains some hyperbolic elements, H called hyperbolic limit.

What is $\overline{\mathrm{Cart}(G)}^{Ch}$ for $n=2$, $n=3$?

- Thm (C.-Leitner-Valette): Up to conjugacy, there is only one limit of $\mathrm{Diag} \leq SL(2, \mathbb{Q}_p)$ in $\overline{\mathrm{Cart}(G)}^{Ch} \setminus \mathrm{Cart}(G)$:

$$H = \left\{ \pm \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{Q}_p \right\} \text{ — unipotent radical} \times \{\pm id\}$$

- **Thm** (C-Leitner-Valette): Up to conjugacy, we have the following limits of Diag in $\overline{\mathrm{Cart}(G)}^{Ch} \setminus \mathrm{Cart}(G)$ for $G = SL(3, \mathbb{Q}_p)$: μ_3 is the group of 3-th roots of unity in $\mathbb{Q}_p^*$.

$$\mu_3 \cdot \begin{pmatrix} a & x & 0 \\ 0 & a & 0 \\ 0 & 0 & \frac{1}{a^2} \end{pmatrix}; \quad \mu_3 \cdot \begin{pmatrix} 1 & x & y \\ 0 & 1 & \alpha x \\ 0 & 0 & 1 \end{pmatrix}; \quad \mu_3 \cdot \begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \mu_3 \cdot \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$$

where $\alpha \in \mathbb{Q}_p$ fixed, $\alpha \in \mathbb{Q}_p^* / \mathbb{Q}_p^{*3}$.

General results about $H \in \overline{\mathrm{Cart}(G)}^{ch} \setminus \mathrm{Cart}(G)$

<u>Thm</u> (C-Leitner-Valette). Up to conjugacy, every hyperbolic limit $H \in \overline{\mathrm{Cart}(G)}^{ch}$ is a subgroup of $B \cap G_{\sigma_+} \cap G_{\sigma_-} = \left\{ \begin{pmatrix} A_1 & & 0 \\ & A_2 & \\ 0 & & \ddots \\ & & & A_k \end{pmatrix} \in SL(n, \mathbb{Q}_p) \right\}$

for some opposite simplices $\sigma_+, \sigma_- \subset \partial\Sigma$ with $\sigma_+ \subset C$, $B = G_C$, and where the blocks $A_1, \dots, A_k$ are indecomposable upper triangular square matrices of possibly different dimensions, and in each block A_i every element in H has its diagonal entries all the same. In particular, H stabilizes a flat of dimension k of Δ and whose ideal boundary is the support of σ_+, σ_-.

<u>Thm</u> (C-Leitner-Valette) (H elliptic limit $\Rightarrow$ unipotent) $SL(n, \mathbb{Q}_p) =: G$
Let $H \in \overline{\mathrm{Cart}(G)}^{ch}$. If H is contained in the Borel subgroup and H does not contain hyperbolic elements, then H is contained in $\mu_n \cdot \begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & \ddots & 1 \end{pmatrix} \to$ unipotent radical of B.
where μ_n is the group of n-th roots of unity in $\mathbb{Q}_p^{*}$.

Thank you!