Aspects of CONNES' EMBEDDING CONJECTURE (PROBLEM)

Discrete groups (countable)

Let Γ be a discrete group

pb. Given $\varepsilon > 0$, $F \subseteq \Gamma$ finite

find N, Φ (depending on ε, F)

$$\Phi : F \longmapsto \mathcal{U}(N)$$

such that

HYPERLINEAR GROUP

1) $\| \Phi(\gamma_1 \gamma_2) - \Phi(\gamma_1)\Phi(\gamma_2) \|_2 < \varepsilon$ if $\gamma_1, \gamma_2, \gamma_1\gamma_2 \in F$

2) $\| \Phi(\gamma) - Id \|_2 > 1 - \varepsilon$ if $\gamma \in F, \gamma \neq e$

Explanation of terms $\mathcal{U}(N) \subseteq M_N(\mathbb{C})$ is the group of unitary matrices

$\|x\|_2^2 = tr(x^* x)$ (making $M_N(\mathbb{C})$ a Hilbert space)

$$tr(A) = \frac{1}{N} Tr(A) = \frac{1}{N} \sum_{i=1}^{N} a_{ii}$$

C. E. Conjecture: All discrete group are hyperlinear

Stronger form. Replacing $\mathcal{U}(M)$ by S_M

S_M group of permutations $\subseteq \mathcal{U}(M)$

we get the definition of sofic groups.

(distance becomes Hamming distance)

Original form of definition of Hyperlinear

Assume for simplicity Γ is finitely generated by $\gamma_1, \gamma_2, \dots \gamma_n$. Then we have a lenght function $|\gamma| = \inf\{s \mid \gamma = \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_s}\}$

If $w = \gamma_{i_1}^{\pm}\gamma_{i_2}\cdots\gamma_{i_p}$ is any word in the generators, then for any M we get a world map (denoted still by w)

$w : \mathcal{U}(M)^n \longrightarrow \mathcal{U}(M)$

$w(U_1, U_2, \dots U_n) = U_{i_1} U_{i_2} \cdots U_{i_p}$

Word maps are object of an intensive study. Questions: When is w surjective

What is w^*(Haar measure)? (ex Puder Magee

it is Haar measure if $\Gamma = F_n$, w a primitive word)

Reformulation of C. E. Conj.:

Given Γ, with generators $\gamma_1, \gamma_2, \ldots \gamma_n$

find, for every $\varepsilon > 0$, every $s_0 \in \mathbb{N}$ (instead of Γ)

N and $U_1, U_2 \ldots U_n$ such that

for all $w = \gamma_{i_1} \ldots \gamma_{i_s}$, $s \le s_0$

$$tr(w(U_1, \ldots U_n)) = \begin{cases} \underset{\varepsilon}{\simeq} 0 \text{ if } w \neq e \\ \underset{\varepsilon}{\simeq} 1 \text{ if } w = e \end{cases}$$

(The equivalence with previous comes from

$\| 1 - u \|_2^2 = 1 - 2tr(u) + 1 = 2(1 - tr u)$)

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Linearization of Γ

$$\mathbb{C}(\Gamma) = \{ \sum_{\gamma \in \Gamma}' a_\gamma \gamma \mid a_\gamma \in \mathbb{C} \}$$

all 0 with exception a finite number

$\mathbb{C}(\Gamma)$ is an algebra: with $\gamma_1 \gamma_2 = (\gamma_1 \gamma_2)$

(product in the algebra = product in the group)

(Ex. If F_2 is the free group with generators a, b then $(a + b + a^{-1} + b^{-1})^2 = \sum_{|w|=2}' w + 4$)

$\mathbb{C}(\Gamma)$ has a trace τ (like trace on matrices)

$$\tau \left(\sum' a_\gamma \gamma \right) = a_e$$

It is a trace ie $\boxed{\tau(xy) = \tau(yx)}$ $(\forall)\, x, y \in \mathbb{C}(\Gamma)$

since $\tau(\gamma_1 \gamma_2) \neq 0 \iff \gamma_1 \gamma_2 = e \iff \gamma_2 \gamma_1 = e \iff \tau(\gamma_2 \gamma_1) = 1$

τ is also positive, ie $\tau(x^* x) \geq 0$

(Since representing $\mathbb{C}(\Gamma) \subseteq B(\ell^2(\Gamma))$ by left convolutions

$$\tau(X) = \langle X \delta_e, \delta_e \rangle \text{ positive functional}$$

In plain words Connes Embedding Conj

says: Approximate the trace on the

group algebra, on finite sets, by

the traces on matrices.

More precisely, General formulation

of Connes Embedding Problem (counterexample

exists)

Given $\mathcal{A}$ $\mathbb{C}$ algebra with involution $*$

(like $\mathbb{C}(\Gamma)$ with $\left(\sum a_g \gamma\right)^* = \sum \bar{a}_g \gamma^{-1}$)

and trace τ (+ boundedness condition
(II_1 von Neumann algeb)

$(\forall) x, (\exists) M$ $\tau(x^* y^* y x) \le M \tau(y^* y)$ $(\forall) y \in \mathcal{A}$)

Given $x_1, x_2, \ldots x_n \in \mathcal{A}$ $(\forall) s_0, \varepsilon > 0$

$(\exists) N \in \mathbb{N}$, $X_1, X_2, \ldots X_n \in M_N(\mathbb{C})$ s.t.

$|\tau(x_{i_1} x_{i_2} \cdots x_{i_s}) - \mathrm{tr}(X_{i_1} \cdots X_{i_s})| < \varepsilon$ for $s \le s_0$

Examples: Case $\mathcal{U} = \mathcal{L}(\Gamma)$

If Γ is residually finite. (i.e. exists $\Gamma_n \triangleleft \Gamma$ (normal of finite index such that $\cap \Gamma_n = e$) then take $\Phi_n : \Gamma \to B(\ell^2(\Gamma/\Gamma_n))$ the representation by left convolution.

They do the job because $(\forall) g \in \Gamma$ there exists a quotient where $\Phi_n(g)$ acts by permutation with no fixed elements, so it has zero trace).

Ex $\Gamma = F_n$ (it is residually finite)
generators $a_1 \dots a_n$

Th. (Voiculescu). If $X_i = \mathrm{Re}\, a_i = \frac{1}{2}(a_i + a_i^*)$

Then random matrices of size N with independent gaussian entries have the property that expected values of traces of monomials in matrices converge to the trace τ of corresponding monomials in the algebra

(7) Voiculescu's Free entropy. = normalized volume (as ...

of microstates (the matrices $X_1, X_2 .. X_n$)

$\chi(X_1, .. X_n) = n$ for $x_1, x_2, .. x_n$ from F_n

...

For C.E. Conjecture : Possible strategy

Take $w_1, w_2 \in F_n$ such that w_2

is not a consequence of w_1 (i.e.

$w_2 \neq$ product of conjugates of w_1)

Obs Then C.E. Conj $\Leftrightarrow$ $(\forall)\varepsilon > 0$ $(\exists)$ $U_1, U_2, .. U_n \in \mathcal{U}(M)$

unitaries such that

$$|\operatorname{tr}(w_1(U_1 .. U_n))| > 1-\varepsilon$$

$$|\operatorname{tr}(w_2(U_1 .. U_n))| < \varepsilon$$

This strategy would lead to

F_n / w_1 is hyperlinear

(relator

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Th (R). The moments

$$\int_{\mathcal{U}(N)^n} tr(\omega_1)^\alpha\, tr(\omega_2)^\beta\, d\eta_{Haar}$$ are

precisely computable if $N > \alpha|\omega_1| + \beta|\omega_2|$

Not sufficient for solving since $tr(\omega)$ are very "small" : $\lim_{N\to\infty} \eta_{\mathcal{U}(N)^n}\left(|tr(\omega(U_1 \dots U_n)| > \varepsilon\right)$

is 0 if ω is nontrivial (Th. Voiculescu)

- Hilbert 17th (non-commutative form)

Given $p(x_1, \dots x_n)$ non-commutative polynomial (assume $x_i = x_i^*$), $p = p^*$

C.E.P $\Leftrightarrow$ If $tr(p(X_1, X_2, \dots X_n)) \geq 0$ (A)

$(\forall)$ $X_1, X_2, \dots X_n$ matrices $\overset{?}{\Rightarrow}$ $\tau(p(x_1, \dots x_n)) \geq 0$ (B)

$(\forall)$ $\mathcal{A}$ with trace, $x_1 \dots x_n \in \mathcal{A}$.

Of course such a p that is of the form $p = \sum_i q_i^* q_i + \sum_j [r_j, a_j]$

has the property B (i.e. $\tau(p) \geq 0$)

Problem (Hilbert) solved by Artin.

Th A positive polynomial is a sum of squares of rationals.

Ex: (commutative variables) (Mokobodzki polynomial)

$$1 + x_1^2 x_2^4 + x_1^4 x_2^2 - 3 x_1^2 x_2^2 =$$

$$= \frac{(x_1^3 x_2 + x_2^3 x_1 - 2 x_1 x_2)^2 (1 + x_1^2 + x_2^2) + (x_1^2 - x_2^2)^2}{(x_1^2 + x_2^2)^2}$$

Th (McCoulough Helton)

If $P(X_1, \dots, X_n) \geq 0$ $(\forall)$ $X_1, \dots X_n$ matrices of controlled size

$\Rightarrow P = \sum_i q_i^* q_i$

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Trace positivity. Th (Procesi + others)

If $\mathrm{tr}(p(X_1, X_2, \dots X_n))$ $(\forall)$ $X_1, X_2, \dots X_n \in M_N(\mathbb{C})$

then $p =$ S.O.S + commutators

modulo the fundamental Cayley Hamilton identity in $M_N(\mathbb{C})$

Th (R). There exists a S.O.S expression for (B) in terms of analytic entire non-commutative series

Many better generalizations (Klepp, ..)

Also (Parillo, others) have SDP methods
M. Laurent

for determining $c \geq 0$ s.t. $p + c$ is a S.OS modulo commutators

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Non-local games formulation (Tsirelson problem)

Heuristically. If two algebras $A, B \subseteq B(H)$ commute: determine how far are these from a situation of the form

$A \subseteq B(H_1)$, $B \subseteq B(H_2)$, $B(H) = B(H_1) \otimes B(H_2)$

(tensor product commutation)

Simplified version. A is generated by $E_1, E_2, \ldots E_n$

B is generated by $F_1, F_2, \ldots F_s$, where $E_1, \ldots E_n$, $F_1, F_2, \ldots F_s$ are projections.

Consider $\mathcal{C}$ (convex set of entangled states

$$\mathcal{C}^{nc} = \left\{ (a_{ij}) = \left(\langle E_i F_j \xi, \xi \rangle \right)_{ij} \;\middle|\; \begin{array}{l} \text{where } A, B \subseteq B(H) \\ \xi \in H,\ \|\xi\| = 1 \end{array} \right\}$$

How far is this from

$\mathcal{C}^{t}$ = same but, $A \subseteq B(H_1)$ $B \subseteq B(H_2)$
$\xi \in H_1 \otimes H_2$

Non-local games strategy (Alice, ...

proved that describing C^{+} would require a Turing machine that solves all halting problems (<u>which is impossible</u>), while C^{uc} can be determined by SDP methods

First part : J. Zhengfeng, A. Vidick, T. Wright, A. Natarajan, H. Yuen in 2020

C^{+} - part solved by Navascues, Pironio

• So $C^{*}(F_n) \otimes C^{*}(F_n)$ has not a unique norm so CEP is disproved.

Many open problems remain

1) C.E. Conjecture for discrete groups

2) Give a positivity criteria for noncommutative polynomials e.g coefficients of t in $\mathrm{Tr}(e^{x+ty})$ if x, y are positive. This is positive on matrices (This is BESSIS-MOUSSA-VILLANI trace conjecture solved by STAHL in 2011